MATHEMATICAL MODEL FOR SMALL ARMS FIRE AGAINST LOW-FLYING AIRCRAFT

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THESIS

MATHEMATICAL MODEL FOR SMALL ARMS FIRE
AGAINST LOW-FLYING AIRCRAFT

by

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20. Abstract (Continued)

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by

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ABSTRACT

Mathematical models for hit probabilities of small arms fire against low-flying aircraft are developed with the aid of the impact points. Three techniques of fire are examined. A model for determining appropriate lead angles is developed. Probability of hit for single shot at various ranges and constant altitude are calculated. Repeated shots are then examined when the technique is used of firing multiple rounds at a fixed angle. The result shows the probability of one of the shots is high, while for other rounds it declines very sharply. Suggestions for further extensions are included.

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I. INTRODUCTION

The problem of computation hit probability for fire against aircraft by small arms organic to the infantry is of increased concern, because of recent emphasis on low altitude air assault tactics.

In firing with small arms against aircraft, there are two types of errors: ballistic and aiming.

1. Ballistic errors relate to characteristics of the projectile and weapon and are caused by variations in projectile weights, variations in weapon barrels, and variation in muzzle velocities.
2. Aiming errors are caused by incorrect pointing of the gun, etc.

For highly time-varying target positions the process of aiming small arms is much more difficult than aiming the gun against stationary targets since the aiming point must be ahead of the target. A model to calculate lead angle at various gun angle of elevation, ballistic and aircraft velocities will be developed in Section II.

These lead angles express the angle by which the gun-bore axis must be deflected from the gun-target line at the instant of fire. By selecting the correct lead angles,

the ballistic can intercept the moving target. There are three techniques of engagement: [1]

1. Pattern of fire technique. The gunner fires continuously at a fixed point on the target path with a pre-selected and fixed aiming angle.
2. Changing lead technique. The gunner extrapolates the target's present line of flight and continuously adjusts his aim point to provide the appropriate lead while firing.
3. Fixed lead. This involves establishing an arbitrary but constant lead along the line of flight and firing continuously.

A basic model for the probability of hit for small-arms fire against low-flying aircraft will be developed in Section III. Only flight path at constant altitude with the approaching aircraft flying directly over the gunner will be considered. The general theoretical formulation involves a density function and some basic geometry which will be used for calculating hit probabilities.

II. BASIC MODEL

A. BASIC MODEL

1. Assumptions for the Theoretical Treatment

In order to be able to treat the hit probability, certain assumptions will be made:

a. The ballistic errors are a random process,

b. The distribution function of the ballistic error is normal,

c. The mean value for the ballistic error is $\mu = 0$,

d. For the two dimensional case $\sigma_x = \sigma_y = \sigma$,

e. The bullet has a flat trajectory along the effective range. (This ignores the effect of gravity on the bullet.)

2. Geometry

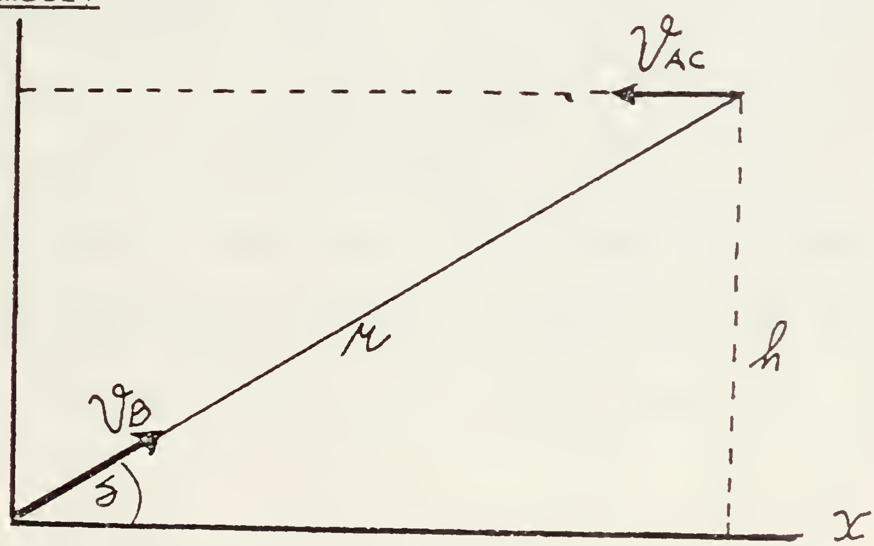

Figure 1: Aircraft Weapon Geometry

Figure 1 illustrates the general situation for firing at targets under the assumption that the aircraft is flying a straight line, unaccelerated course.

Since we do not account for the effect of gravity, the trajectory of the bullet is a straight line which will intersect with the line of the bullet's trajectory. The angle that the bullet's trajectory makes with the horizontal is the initial angle of inclination of the trajectory and is also called the angle of departure.

If we only consider the projection on the plane (X,Z) containing both aircraft and bullet trajectories the following quantities may be used in developing the model in a one dimensional case:

x = horizontal range to target,

h = altitude of the target (aircraft),

g = initial angle of inclination of gun,

GF = slant range (r),

B = velocity of ballistic, and

AC = velocity of the aircraft

and the symbols G for gun, F for target (aircraft). For the two dimensional case the trajectory of the bullet also may have a projection in the (X,Y) plane.

B. FORMULATION OF LEAD ANGLE

In the hit problem, we will be concerned with the problem of successfully aiming the gun so a bullet will intercept a moving target. If the gun is pointed directly at the target at the time the gunner fires, he will make an error because of the target's motion during the projectile's time in flight.

It is important to develop a formula expressing the angle by which the gun-bore axis must be deflected from the gun target line at the instant of fire. This angle is called "the lead angle."

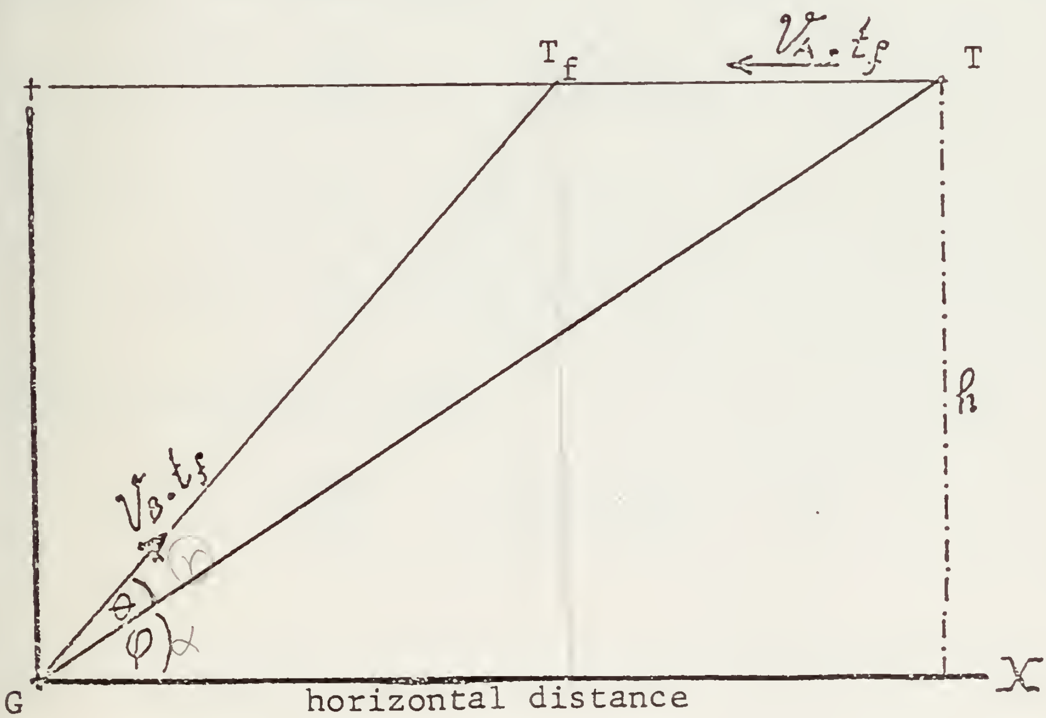

Figure 2: Lead Angle

T = target

T_f = future position of target

θ = lead angle

V_A = velocity of aircraft

V_B = velocity of bullet

GT = gun target line

In this model we will only consider the kinematic lead, and ignore the gravity since the ballistics of small arms have an essentially flat trajectory within the effective range of fire.

Suppose the direction of the bullet when fired is GT_f, bullet velocity V_B and time of aircraft flight t_f. From the triangle $GT\,T_f$ and the law of sines yield:

$$\frac{\sin\theta}{T\,T_f} = \frac{\sin\phi}{G\,T_f}$$

$$\sin\theta = \frac{T\,T_f}{GT_f} \cdot \sin\phi = \frac{V_A\ t_f}{V_B\ t_f} \cdot \sin\phi$$

$$\sin\phi = \frac{h}{GT} = \frac{h}{\sqrt{(x^2+h^2)}}$$

$$\sin\theta = \frac{V_A}{V_B} \cdot \frac{h}{\sqrt{(x^2+h^2)}}$$

$$\theta = \sin^{-1} \frac{V_A\ h}{V_B\sqrt{(x^2+h^2)}} \qquad (1)$$

or if lead angle is small

$$\theta = \frac{V_A}{V_B} \cdot \frac{h}{\sqrt{(x^2+h^2)}} \quad \text{rad} \qquad (2)$$

For illustration, Table 1 shows the required lead angle θ in degrees and in radians for an aircraft with velocity 100,200,400 knots with altitude 100m, 150m, 200m, 250m, 300m, 350m, 400m, when a small arm with bullet velocity

2750 ft/sec is fired at a horizontal distance of 350 meters. Table 2 shows the result of θ in degrees and in radians from the same ballistic and the same aircraft as in Table 1, but with altitude constant at 100m.

The horizontal range changes from 0 to 400m with 25m increments.

Figure 3 shows the relationships between θ and altitude, if velocity of the aircraft and the bullet are known.

Figure 4 shows the relationship between θ and the horizontal range, for three velocities of the aircraft at an altitude of 100 meters.

Table 1

The Value of Angle θ
Corresponds to the Velocity & Altitude
of an Aircraft

ALTI-TUDE	V_{AC} = 100 KNOTS		V_{AC} = 200 KNOTS		V_{AC} = 400 KNOTS	
	ANGLE θ IN		ANGLE θ IN		ANGLE θ IN	
	RADIANS	DEGREES	RADIANS	DEGREES	RADIANS	DEGREES
100m	.0168	.962	.0336	1.925	.0672	3.8531
150m	.0241	1.383	.0482	2.762	.0964	5.5319
200m	.0304	1.742	.0608	3.485	.1216	6.9844
250m	.0356	2.041	.0712	4.082	.1424	8.1867
300m	.0398	2.286	.0796	4.565	.1592	9.1604
350m	.0433	2.484	.0866	4.968	.1732	9.9739
400m	.0461	2.644	.0922	5.290	.1844	10.6261

NOTE: X = 350m
V_B = 2750 FT/sec

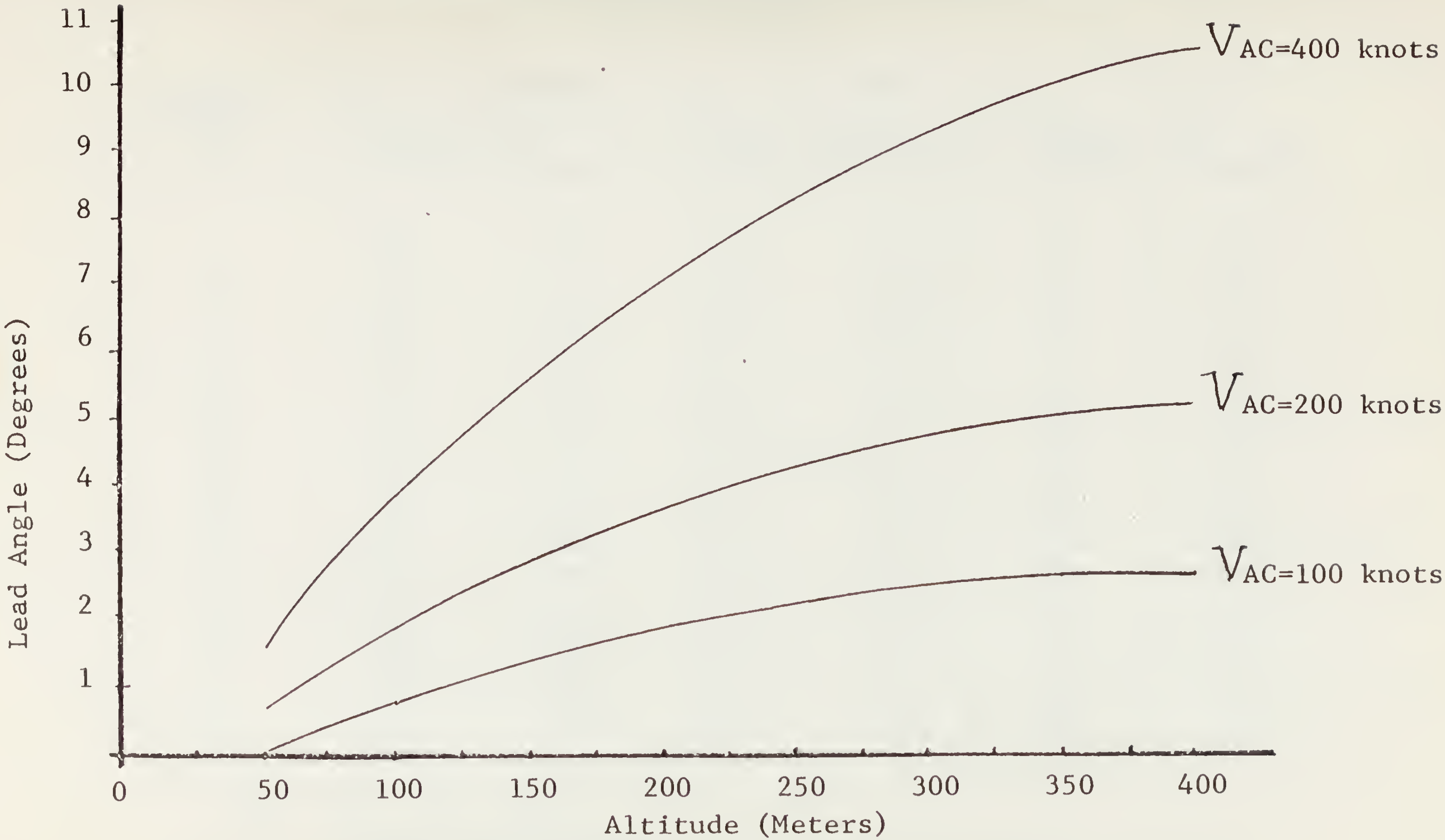

Figure 3: Lead Angle vs Altitude in Various Aircraft's Velocity

Table 2
The Value of Angle θ
Corresponds to the Velocity and Horizontal Ranges
of an Aircraft

HORIZ. RANGE (X)	V_{AC} = 100 KNOTS		V_{AC} = 200 KNOTS		V_{AC} = 400 KNOTS	
	ANGLE θ IN		ANGLE θ IN		ANGLE θ IN	
	RADIANS	DEGREES	RADIANS	DEGREES	RADIANS	DEGREES
0	.0613	3.514	.1226	7.0421	.2452	14.1936
25	.0594	3.409	.1188	6.8228	.2376	13.7449
50	.0548	3.143	.1096	6.2922	.2192	12.6620
75	.0490	2.810	.0980	5.6240	.1960	11.3031
100	.0433	2.484	.0866	4.9680	.1732	9.9739
125	.0382	2.194	.0764	4.3816	.1529	8.7892
150	.0340	1.948	.0680	3.8991	.1360	7.8164
175	.0304	1.742	.0608	3.4857	.1260	6.9844
200	.0274	1.570	.0548	3.1413	.1096	6.2922
225	.0248	1.426	.0496	2.8430	.0992	5.6931
250	.0227	1.304	.0454	2.6021	.0908	5.2096
275	.0209	1.200	.0418	2.3956	.0836	4.7955
300	.0193	1.110	.0386	2.2121	.0772	4.4276
325	.0180	1.032	.0360	2.0630	.0720	4.1288
350	.0168	.964	.0336	1.9255	.0672	3.8531
375	.0157	.905	.0314	1.7993	.0628	3.6005
400	.0148	.851	.0296	1.6962	.0592	3.3938

NOTE ALTITUDE: 100m
V_B + 2750 FT/sec
1 KNOT = 1.85 km/HR

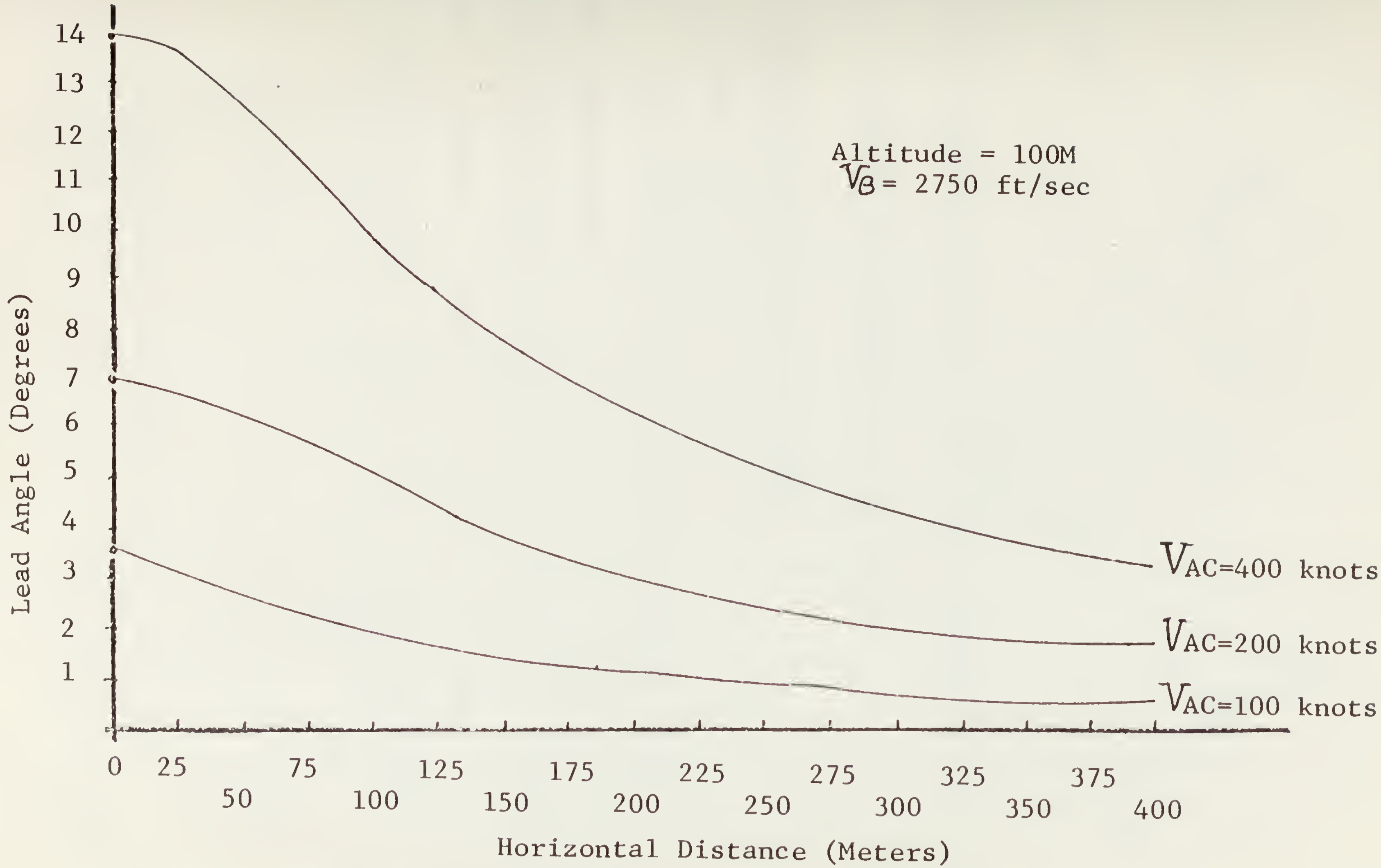

Figure 4: Lead Angle vs Horizontal Distance in Various Aircraft's Velocity

. OTHER GEOMETRY

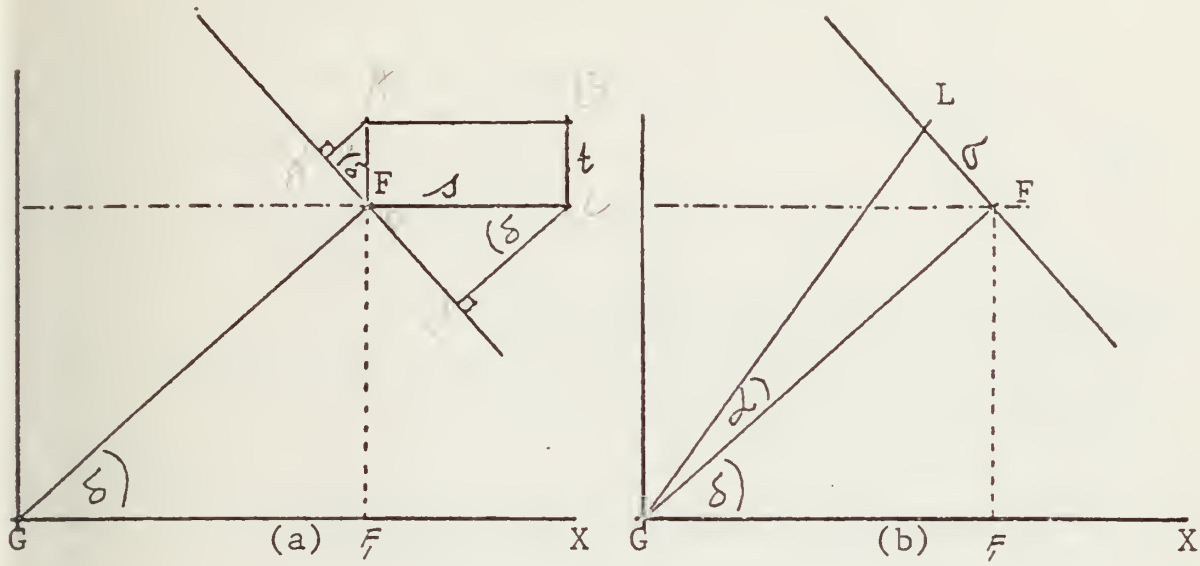

Figure 5: Geometry of the Target

1. Aircraft Presented Projection

a. One Dimensional Case

For practical case of firing against aerial :argets we are confronted with complicated target shapes. 4]. In the one dimensional case we consider the aircraft ıs a two rectangular dimensional target. The impact point of the target is normally distributed on the line perpendicular to the ballistic trajectory (gun-target line). Let s be the length of the aircraft and t be the height of the aircraft. From Figure 5 (a), the projection of s to the line perpendicular GUN-TARGET line is $s \sin \delta$ and the projection of t is $t \cos \delta$. Then the projection

of the aircraft to the line perpendicular bullet trajectory can be approximately by the equation

$$\ell = s \sin \delta + t \cos \delta . \tag{3}$$

b. Two Dimensional Case

As in the one dimensional case, the aircraft can be projected to the plane perpendicular to the trajectory.

Let A_1 be the horizontal area

A_2 be the vertical area

The projection of A_1 to that plane is $A_1 \sin \delta$ and the projection of A_2 is $A_2 \cos \delta$.

Then the projection of the aircraft to the plane perpendicular to the bullet trajectory can be approximately by the equation

$$A = A_1 \sin \delta + A_2 \cos \delta \tag{4}$$

2. <u>Formulation of Ballistic Dispersion (σ)</u>

Let α be the dispersion angle of the bullet known for a certain type of gun or bullet. From the triangle GFL as shown in Figure 5 (b), GF perpendicular to FL, then $\frac{\sigma}{GF} = \tan \alpha$ or $\sigma = GF \tan \alpha$.

From the triangle GFF_1 · $GF = h/\sin \delta$ then $\sigma = \dfrac{h \tan \alpha}{\sin \delta}$

(5)

for illustration, Table 3 shows the value of ℓ and σ for various distance with constant altitude of 100 meters where the length of the aircraft is 8 meters and the width of the aircraft = 2 meters.

Table 3

Relation Between δ, σ, ℓ

X	$\delta = \tan^{-1} h/x$	$\sigma = \frac{h \tan\alpha}{\sin\delta}$	$\ell = 8\sin\delta + 2\cos\delta$
400	26.57	3.9035	5.3671
350	29.74	3.5197	5.7050
300	33.69	3.1465	6.1016
250	38.66	2.7949	6.5593
200	45.00	2.4692	7.0710
150	53.13	2.1825	7.5999
100	63.43	1.9521	8.0490
50	75.96	1.7997	8.2462
0	90.00	1.7460	8.0000

Note: h=200 meters; $\tan\alpha$ = .00873

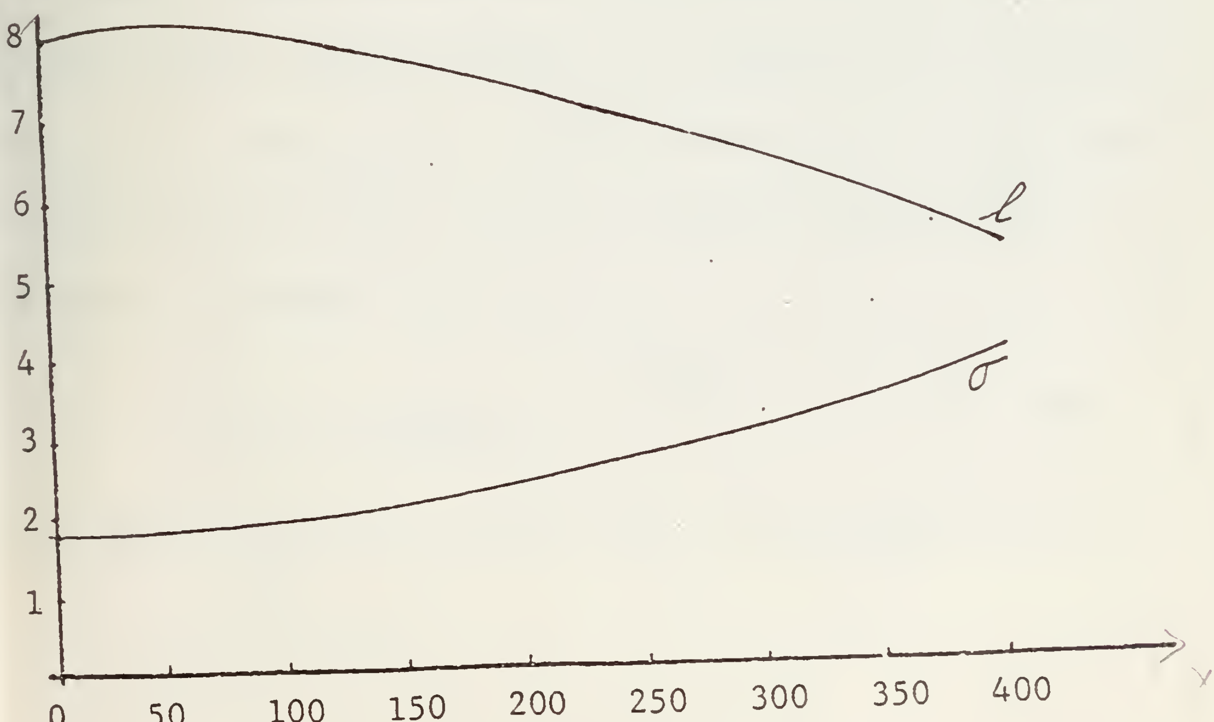

Figure 6: Chart of ℓ and σ

III. ONE DIMENSIONAL CASE

A. SINGLE SHOT

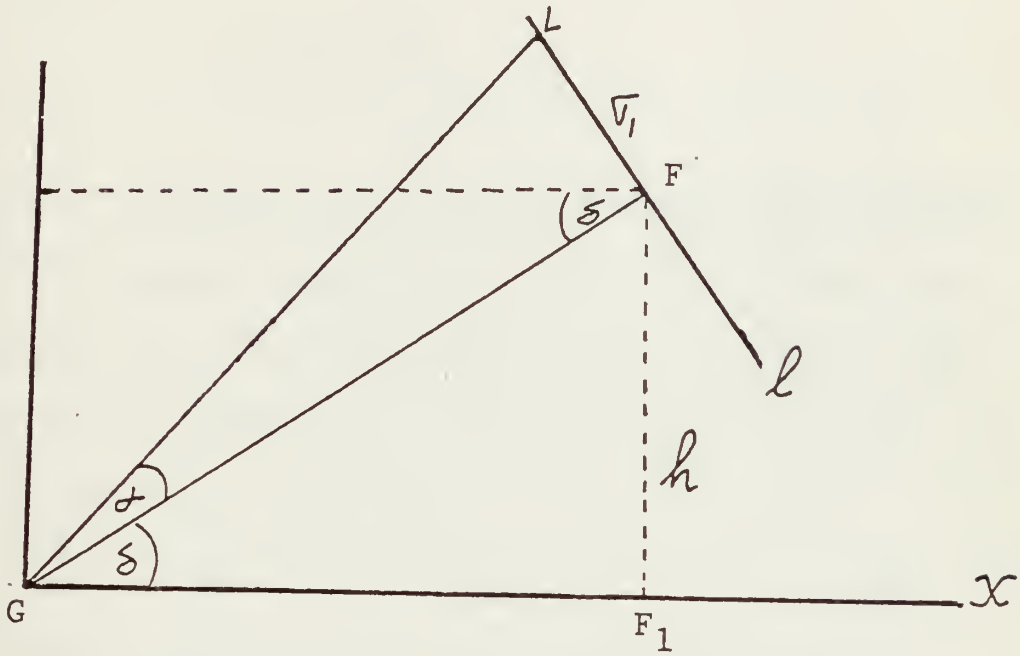

Figure 7: Geometry of the Target at Single Shot

Let the impact point of the bullet be a random variable X, normally distributed on the line ℓ perpendicular to the GUN-TARGET line GF (Figure 7). Assuming the distribution of the point of the bullets intersection of the line is centered at F ($\mu = 0$) with standard deviation σ_1, the density distribution of X is:

$$f(x) = \frac{1}{\sigma_1 \sqrt{2\pi}} \cdot e^{-x^2/2\sigma_1^2} \qquad (6)$$

The probability of hit in the line ℓ centered at F is:

$$\int_{-\ell/2}^{\ell/2} f(x)\, dx$$

Then: $\quad Pr\text{ (hit)} = Pr\left(-\ell/2 \leq X \leq \ell/2\right)$

$$= \Phi\left(\frac{\ell/2}{\sigma_1}\right) - \Phi\left(-\frac{\ell/2}{\sigma_1}\right)$$

$$= 2\Phi\left(\frac{\ell}{2\sigma_1}\right) - 1 \tag{7}$$

where $\Phi(a) = \text{Prob}(x \leq a)$ can be obtained from standard tables for the normal distribution. Let s be the length of the aircraft and t be the height of the aircraft. The projection of the aircraft to the bullet trajectory can be approximated by the equation:

$$\ell = s \sin\delta + t \cos\delta \tag{8}$$

Substituting equation (8) into equation (7) yields:

$$Pr\text{ (a hit)} = 2\Phi\left(\frac{s \sin\delta + t\cos\delta}{2\sigma_1}\right) - 1 \tag{9}$$

as calculated in Section IIC, $\sigma_1 = \dfrac{h \tan\alpha}{\sin\delta}$

Substituting the value of σ_1 into equation (9) yields:

$$Pr\text{ (a hit)} = 2\Phi\left\{\frac{(s\sin\delta + t\cos\delta)\sin\delta}{2\,h\tan\alpha}\right\} \tag{10}$$

Example: An aircraft with velocity 100 knots is fired upon by a rifle, the velocity of the bullet V_B = 2750 ft/sec = 838.2 meter/sec. The aircraft has a constant altitude 200m and its horizontal distance from aircraft to the firer = 200m let $\alpha = 0.5^\circ$ be the dispersion angle of the bullet. We have $\phi = \arctan\frac{250}{200} = 51.3401$. lead angle θ as shown in Table 2. For this geometry θ=1.570, then

$\delta = \varphi + \theta = 51.3401 + 1.570 = 52.9101$. Let the length of the aircraft be $\mathcal{s} = 8$ meters and the height of the aircraft be t = 2 meters, then:

$$\Pr(\text{a hit}) = 2\,\Phi\left\{\frac{(\ \sin\delta + t\cos\delta)\ \sin\delta}{2\ h\ \tan\alpha}\right\} -1$$

$$= 2\,\Phi\left\{\frac{8\ \sin 52.9101 + 2\ \cos 52.9101)\ \sin 52.9101}{2 \times 250\ \tan 0.5}\right\}$$

$$= 2\,\Phi\ \left(\frac{6.05259}{4.3634}\right)\ -1$$

$$= 2\,\Phi\ 1.38\ -1$$

$$= 0.8324$$

B. MULTIPLE SHOTS

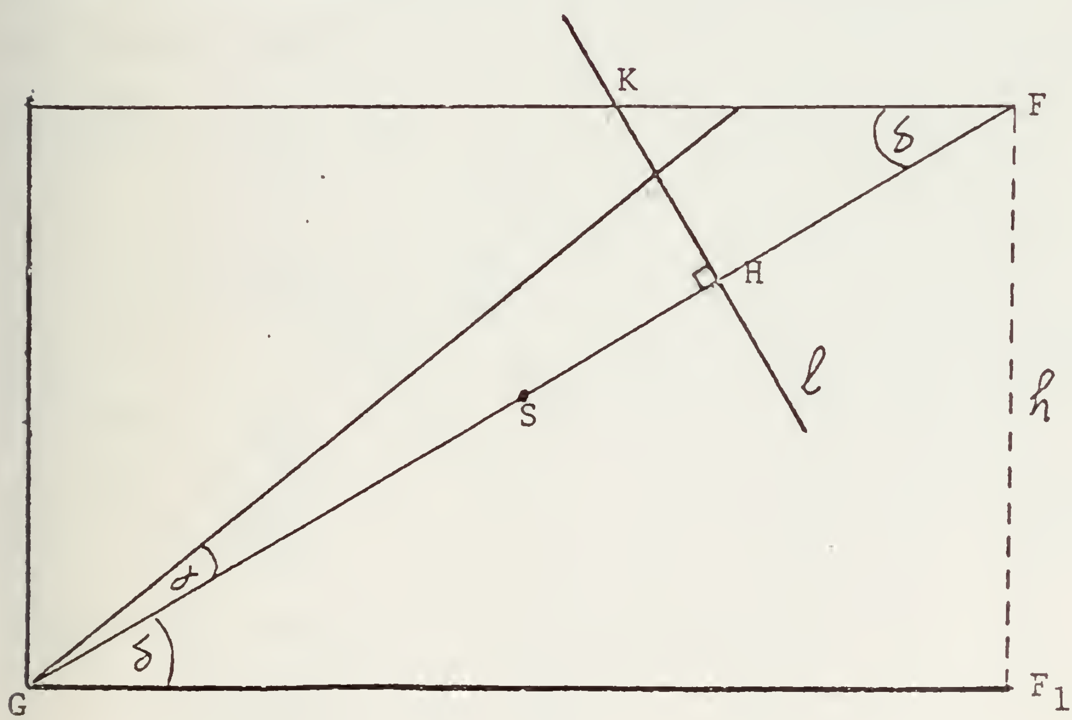

Figure 8: Geometry of the Target for Multiple Shots

EVALUATION OF THE SECOND ROUND OF FIRE

1. MODE 1: Pattern of Fire

Here the gun has the same position as the first round (see Figure 8). Let the difference time between the first bullet and the second bullet = Δt. The first bullet will intersect the line of the aircraft (the line perpendicular to GF at the point F at time t). At time t, the second bullet is at the point S. Suppose the second bullet intersects the line perpendicular to GF at the point H at time $t+t_1$ and the airplane has moved from F to the point K. The time for the aircraft to travel from F to K must equal time for the second bullet to travel from S to H,

$$FH = FS - SH$$

$$= V_B \Delta t - V_B t_1$$

$$t_1 = \frac{V_B \Delta t - FH}{V_B} \tag{11}$$

$$FK = V_{AC} \cdot t_1$$

$$t_1 = \frac{FK}{V_{AC}} \tag{12}$$

accordingly setting (11) equal to (12) is

$$t_1 = \frac{V_B \Delta t - FH}{V_B} = \frac{FK}{V_{AC}} \quad ;$$

but

$$FH = FK \cos\delta \quad \text{so } t_1 = \frac{V_B \cdot \Delta t - FK \cos\delta}{V_B} = \frac{FK}{V_{AC}}$$

or

$$V_B \cdot V_{AC} \cdot \Delta t - FK\, V_{AC} \cos\delta = FK \cdot V_B$$

$$FK \cdot V_B + FK\, V_{AC} \cos\delta = V_B \cdot V_{AC} \cdot \Delta t$$

$$FK\,(V_B + V_{AC} \cos\delta) = V_B \cdot V_{AC} \cdot \Delta t$$

$$FK = \frac{V_B \cdot V_{AC} \cdot \Delta t}{(V_B + V_{AC} \cos\delta)}$$

$$KH = FK \sin\delta = \frac{V_B \cdot V_{AC} \cdot \Delta t \sin\delta}{V_B + V_{AC} \cos\delta} \qquad (13)$$

$$\Pr(\text{second hit}) = \int_{KH-\ell/2}^{KH+\ell/2} f(x)\,dx = \Phi\left(\frac{KH+\ell/2}{\sigma_2}\right) - \Phi\left(\frac{KH-\ell/2}{\sigma_2}\right) \qquad (14)$$

where $\ell = s \sin\delta + t \cos\delta$ and $\sigma_2 = \dfrac{h \tan\alpha}{\sin\delta}$

2. MODE 2: Changing Lead

With a changing lead, the direction of the gun moves to the new position with the correct lead angle for the newly positioned target. Let the lead angle for that new position be γ (as calculated at Table 2). Then the new firing angle is $\delta + \gamma$, and

$$\Pr(\text{second hit}) = \int_{-\ell/2}^{\ell/2} f(x)\,dx = 2\,\Phi\left(\frac{\ell}{2\sigma}\right) - 1$$

where $\ell = s \sin(\delta + \gamma) + t \cos(\delta + \gamma)$

and $\sigma = \dfrac{h \tan\alpha}{\sin(\delta + \gamma)}$

Thus,

Pr (second hit) =

$$2\Phi\left[\frac{\{s \sin(\delta+\gamma) + t\cos(\delta+\gamma)\}\sin(\delta+\gamma)}{2h\tan\alpha} - 1\right] \quad (15)$$

Results for probability of hit at various ranges without lead angle and with correct lead angle are shown in Tables 4 and 5 and the accompanying graphs are shown in Figures 9 and 10.

3. MODE 3: Fixed Lead

Let φ be the constant lead angle selected from the set of leads appropriate for target speeds and target gunner distances. Then the new firing angle is $\delta + \varphi$.

$$\text{Pr (second hit)} = \int_{-\ell/2}^{\ell/2} f(x)$$

$$= 2\Phi\left(\frac{\ell}{2\sigma}\right) - 1$$

when $\ell = s\sin(\delta+\varphi) + t\cos(\delta+\varphi)$

and $\sigma = \dfrac{h\tan\alpha}{\sin(\delta+\varphi)}$

Pr (second hit) =

$$2\Phi\left[\frac{\{s\sin(\delta+\varphi) + t\cos(\delta+\varphi)\}\sin(\delta+\varphi)}{2h\tan\alpha} - 1\right] \quad (16)$$

Table 4

Probability A Hit at Various Ranges Without Lead Angle

x	δ	$\sigma = \frac{h \tan\alpha}{\sin\delta}$	ℓ	Pr (Hit)
400	14.0362	3.5981	3.8805	.4038
350	15.9453	3.1766	4.1208	.4778
300	18.4349	2.7596	4.4271	.5762
250	21.8014	2.3497	4.8280	.6922
200	26.5650	1.9513	5.3665	.8354
150	33.6900	1.5732	6.1016	.9464
100	45.	1.2341	7.0710	.9958
50	63.4349	.9756	8.0498	1.
0	90	.8726	8.	1.

Table 5

Probability A Hit at Various Ranges With Correct Lead Angle

x	$\delta + \theta$	$\sigma = \frac{h \tan\alpha}{\sin(\delta+\theta)}$	ℓ	Pr (Hit)
400	15.7324	3.2185	4.0942	.4714
350	17.8709	2.8438	4.3585	.5528
300	20.6470	2.4749	4.6924	.6528
250	24.4035	2.1122	5.1265	.7738
200	29.7063	1.7610	5.7015	.8926
150	37.5892	1.4306	6.4647	.9756
100	49.9680	1.1397	7.4119	.9988
50	69.7271	.9303	8.1974	1.0
0	97.0421	.8793	7.6944	1.0

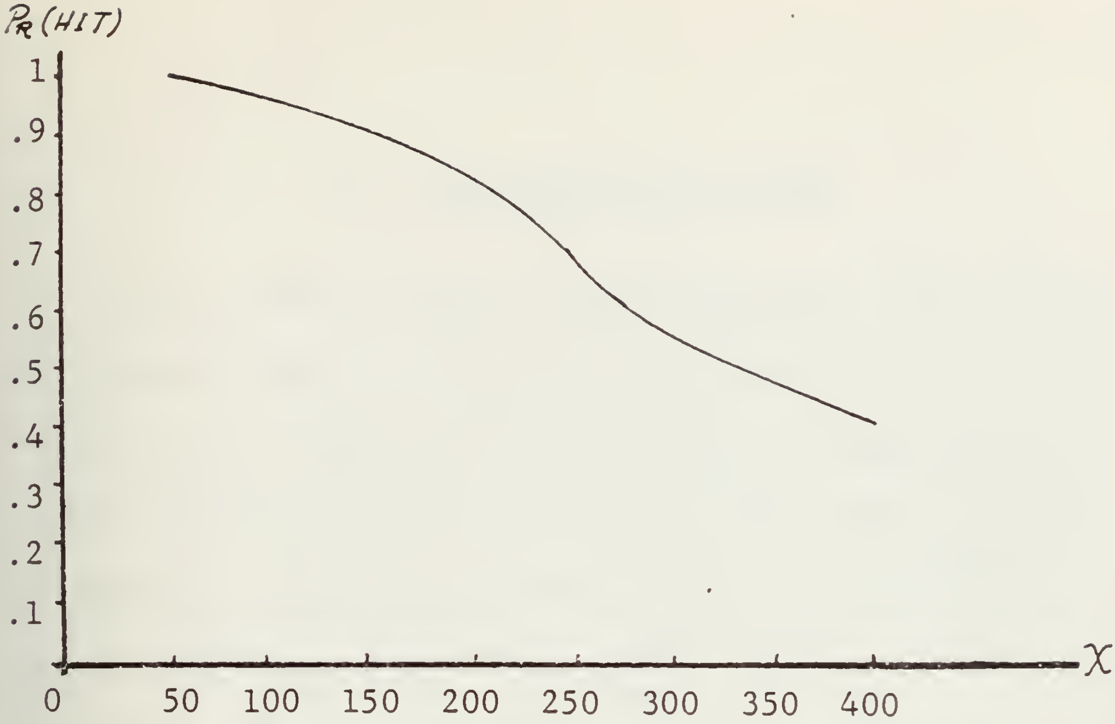

Figure 9: Probability a Hit at Various Ranges Without Lead Angle

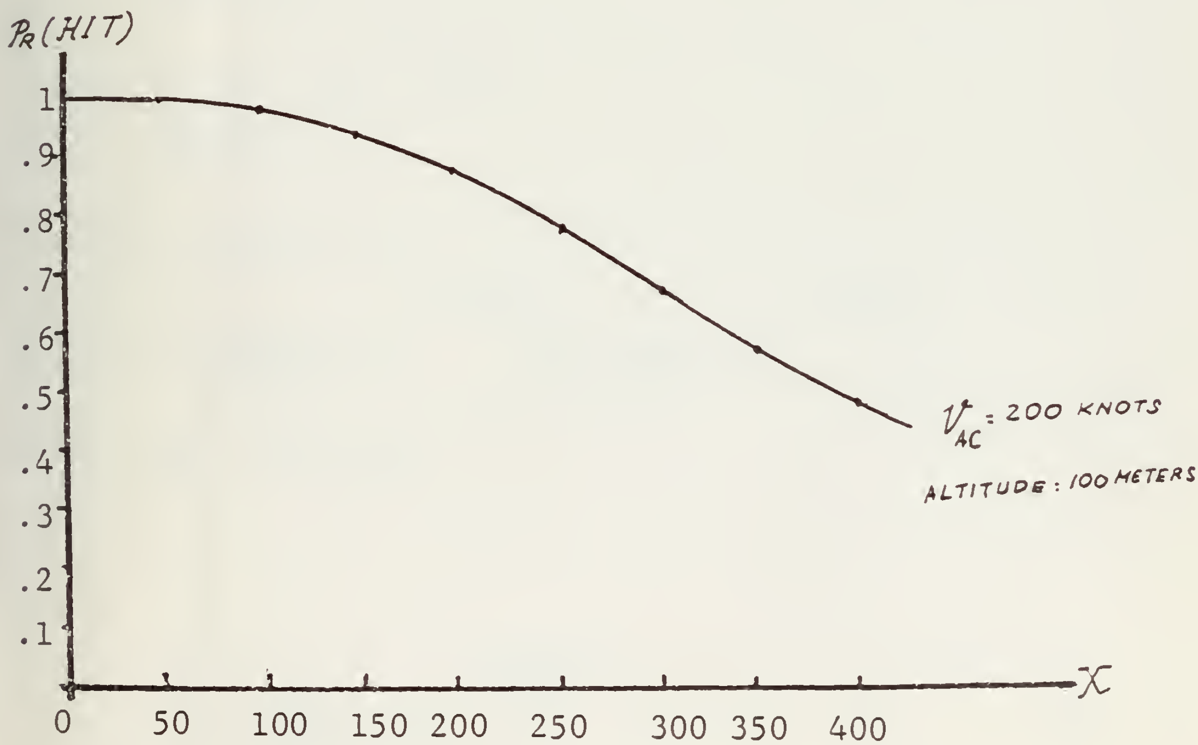

Figure 10: Probability a Hit at Various Ranges With Correct Lead Angle

IV. TWO DIMENSIONAL CASE

Let the impact point of the bullet on a plane which is perpendicular to the bullet trajectory and which coincides with the center of the aircraft be represented by random variables X and Y. It is assumed that X, Y are independently and normally distributed $N(X;\ 0;\ \sigma_x^2)$, $N(Y;\ 0;\ \sigma_y^2)$ and $\sigma_x^2 = \sigma_y^2 = \sigma^2$.

The density functions of X and Y are:

$$f_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} \cdot e^{-X^2/2\sigma_X^2}$$

$$f_Y(Y) = \frac{1}{\sigma_Y \sqrt{2\pi}} \cdot e^{-Y^2/2\sigma_Y^2}$$

The joint density is:

$$f_{XY}(X,Y) = \left(\frac{1}{\sigma_X \sqrt{2\pi}} \cdot e^{-X^2/2\sigma_X^2}\right)\left(\frac{1}{\sigma_Y \sqrt{2\pi}} \cdot e^{-Y^2/2\sigma_Y^2}\right)$$

If $\sigma_X = \sigma_Y = \sigma$ then:

$$f_{XY}(X,Y) = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{X^2+Y^2}{2\sigma^2}} \tag{17}$$

The probability of a hit in an area A is:

$$\text{Pr (a hit)} = \int_A\int \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{X^2+Y^2}{2\sigma^2}}\, dX\, dY$$

Assume that the target is circular with radius R. transforming to the polar coordinates:

$$X = r\cos\theta \quad , \quad Y = r\sin\theta$$

$$X^2 + Y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$dX\,dY = r\,dr\,d\theta$$

$$\Pr(\text{a hit}) = \frac{1}{2\pi\sigma^2}\int_0^R\int_{-\pi}^{\pi} r\,e^{-r^2/2\sigma^2}\,d\theta\,dr$$

$$= \frac{2\pi}{2\pi\sigma^2}\int_0^R r\,e^{-r^2/2\sigma^2}\,dr$$

$$= -e^{-r^2/2\sigma^2}\Big]_0^R$$

$$\Pr(\text{a hit}) = 1 - e^{-R^2/2\sigma^2} \tag{18}$$

Recall the series expansion of

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots\cdots$$

If X is small, the second and higher order terms in the series can be ignored,

i.e.: $e^X \approx 1 + X$

From the formula (18) if R is small,

$$e^{-R^2/2\sigma^2} = 1 - R^2/2\sigma^2 , \qquad \text{then}$$

$$\begin{aligned} \Pr\text{ (hit)} &= 1 - e^{-R^2/2\sigma^2} \\ &= 1 - 1 + R^2/2\sigma^2 \\ &= R^2/2\sigma^2 \end{aligned}$$

Since the area of the target is πR^2

$$\Pr\text{ (a hit)} = R^2/2\sigma^2 = \frac{\pi}{\pi} \cdot R^2/2\sigma^2 = \frac{A}{2\pi\sigma^2}$$

Accordingly for a small target (if $R < \sigma$),

$$\text{then} \quad \Pr\text{ (hit)} = \frac{A}{2\pi\sigma^2} \qquad (19)$$

While for the large target ($R > \sigma$),

$$\Pr\text{ (hit)} = 1 - e^{\frac{A}{2\pi\sigma^2}} \qquad (20)$$

From section IIC:

$$A = A_1 \cos\gamma + A_2 \sin\gamma$$

$$\sigma = \sigma_1 = \frac{h \tan\alpha}{\sin\delta}$$

Substituting the value of A and σ to the equation (19) yields:

$$\Pr\text{ (hit)} = \frac{A_1 \cos\gamma + A_2 \sin\gamma}{2\pi\left(\frac{h \tan\alpha}{\sin\delta}\right)^2} \qquad (21)$$

(i.e. the probability of a hit for small target)

and if the target is large:

$$\Pr(\text{hit}) = 1 - e^{\frac{A_1 \cos\gamma + A_2 \sin\gamma}{2\pi\left(\frac{h\tan\alpha}{\sin\delta}\right)^2}} \tag{22}$$

For an illustration of the application of these models, we consider the following example:

Given: V_B = velocity of the bullet = 710m

h = aircraft altitude = 200m

X = horizontal range from gun to aircraft = 300m

V_{AC} = velocity of aircraft = 300 knots = 154.164 km/sec

$\sigma_\alpha = 0.5°$

rate of fire = 600 rounds/minutes

Δt = time between rounds

= .1 second

s = length of the aircraft

= 8m

t = height of the aircraft

= 2m

δ = arc tan 200/300 = 33.69

ℓ = 8 sin 33.69 + 2 cos 33.69 = 6.1016

$\ell/2$ = 3.050

$$\sigma = \frac{200 \tan 0.5}{33.69} = 3.146$$

The values of ℓ and σ are assumed constant for the specified angle δ .

Instead of using the lead angle we shift the position of aircraft $k\Delta t$ forward from the initial hit position ($k = 0.1, 0.2, . . 1.0$). The result is shown in Table 6.

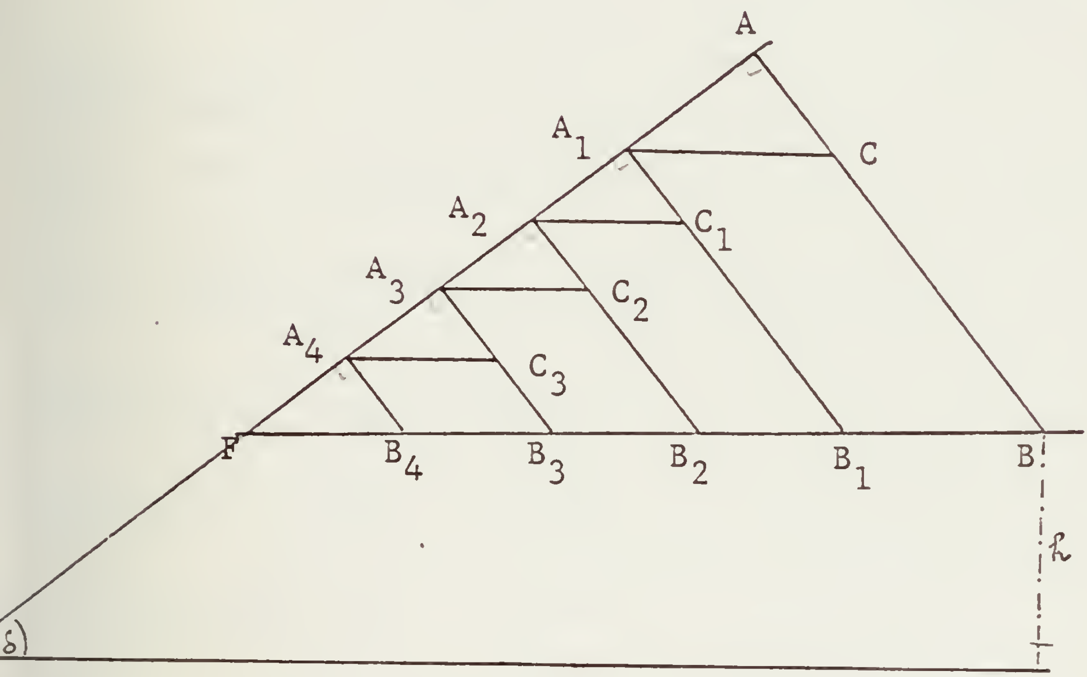

Figure 11: Dynamics of Impact Points

<u>First Round</u>

$FB = 2 \Delta t \, V_{AC}$

$FA = FB \cos \delta = 2 \Delta t \, V_{AC} \cos \delta$

$FA_1 = V_B \, t_1$

$A_1A = FA - FA_1 = 2 \Delta t \, V_{AC} \cos \delta - V_B \, t_1$

$$A_1C = \frac{A_1A}{\cos\delta} = \frac{2\Delta t\, V_{AC}\cos\delta - V_B\, t_1}{\cos\delta}$$

$$A_1C = B_1B = V_{AC}\, t_1$$

$$t_1 = \frac{2\Delta t\, V_{AC}\cos\delta}{V_{AC}\cos\delta + V_B}$$

$$\Pr(\text{Hit}) = \Phi\frac{(A_1B_1 + \ell/2)}{\sigma} - \Phi\frac{(A_1B_1 - \ell/2)}{\sigma}$$

<u>Second Round</u>

$$A_2A_1 = (\Delta t - t_2)\, V_B$$

$$A_2C_1 = \frac{(\Delta t - t_2)\, V_B}{\cos\delta} = B_1B_2 = V_{AC}\, t_2$$

$$(\Delta t - t_2)V_B = V_{AC}\, t_2 \cos\delta$$

$$V_{AC}t_2\cos\delta + V_Bt_2 = V_B\Delta t$$

$$t_2\,(V_{AC}\cos\delta + V_B) = V_B\,\Delta t$$

$$t_2 = \frac{V_B\,\Delta t}{V_B + V_{AC}\cos\delta}$$

$$\Pr(\text{Hit}) = \Phi\frac{(A_2B_2 + \ell/2)}{\sigma} - \Phi\frac{(A_2B_2 - \ell/2)}{\sigma}$$

<u>Third Round</u>

$$A_3A_2 = (\Delta t - t_3)\, V_B$$

$$A_3C_2 = \frac{(\Delta t - t_3)V_B}{\cos\delta} = B_3B_2 = V_{AC}t_3$$

$$(\Delta t - t_3)V_B = V_{AC}\, t_3 \cos\delta$$

$$V_B t_3 + V_{AC} t_3 \cos\delta = \Delta t\, V_B$$

$$t_3\,(V_B + V_{AC}\cos\delta) = \Delta t\, V_B$$

$$t_3 = \frac{\Delta t\, V_B}{V_B + V_{AC}\cos\delta}$$

$$\Pr(\text{Hit}) = \Phi\,\frac{(A_3B_3 + \ell/2)}{\sigma} - \Phi\,\frac{(A_3B_3 - \ell/2)}{\sigma}$$

By putting the numerical example to these calculation we get the results:

$t_1 = 0.030$

$FA_1 = V_B t_1 = 7.0 \times 0.030 = 21.3$

$A_1B_1 = 21.3 \tan 33.69 = 14.199$

$\Pr(\text{Hit}) = \Phi(15.482) - \Phi(3.543)$

$= \underline{.0002}$

$t_2 = 0.084$

$FB_2 = V_{AC}\,(2\Delta t - t_1 - t_2)$

$= 13.258$

$A_2B_2 = FB_2 \sin 33.69 = 7.3542$

$\Pr(\text{Hit}) = \Phi\, 3.3071 - \Phi\, 1.3681$

$= 0.0864$

$t_3 = t_2 = 0.084$

$FB_3 = FB_1 - 2V_{AC}t_2 = 0.03083$

$A_3B_3 = FB_3 \sin 33.69 = .01710$

$\Pr(\text{Hit}) = \Phi .9749 - \Phi - .9658$

$= .6655$

$t_4 = t_3 = 0.0864$

$FB_4 = FB_1 - 3V_{Ac}\, t_2 = - 12.64144$

$A_4B_4 = - 7.0122$

$\Pr(\text{Hit}) = \Phi - 1.2594 - \Phi(- 3.1984)$

$= .1049$

From Table 6 we can calculate Pr (n Hits) is

$$\Pr(1\ \text{Hit}) = \sum_{i=1}^{4} P(H_i) \prod_{j \neq i} P(M_j), \text{ where } P(M_j) = 1 - P(H_j)$$

Then:

$\Pr(1\ \text{Hit}) = .5728$

$$\Pr(2\ \text{Hits}) = \sum_{i=1}^{4} P(H_i) \sum_{j \neq i} P(H_j) \prod_{k \neq i,j} (M_k)$$

$= .1236$

$\Pr(3\ \text{Hits}) = .00605$

$\Pr(4\ \text{Hits}) = .000001$

Results for probability of his of 4 rounds in sequence with $k\Delta t$ increments are shown in Table 6. The expected value of hits with respect to Δt is shown in Table 7 and the accompanying graph is shown in Figure 12.

Table 6

Probability of Hit of 4 Rounds in Sequence
With *k Δt Increment

INCREMENT	$0.0\Delta t$	$0.1\Delta t$	$0.2\Delta t$	$0.3\Delta t$	$0.4\Delta t$	$0.5\Delta t$	$0.6\Delta t$	$0.7\Delta t$	$0.8\Delta t$	$0.9\Delta t$	$1.0\Delta t$
PROB / SHOT #	PH(0)	PH(1)	PH(2)	PH(3)	PH(4)	PH(5)	PH(6)	PH(7)	PH(8)	PH(9)	PH(10)
1	.0002	.0004	.0010	.0023	.0052	.0110	.0217	.0408	.0705	.1143	.1741
2	.0864	.1367	.2032	.2848	.3761	.4738	.5551	.6211	.6589	.6628	.6305
3	.6655	.6544	.6109	.5375	.4534	.3591	.2688	.1897	.1241	.0774	.0454
4	.1049	.0627	.0358	.0192	.0139	.0047	.0019	.0008	.0003	.0002	.0001
Pr (Hit)	.8570	.8542	.8509	.8438	.8486	.8386	.8475	.8524	.8538	.8547	.8501

* k = 0.0, 0.1, 0.2, ------ 0.9, 1.0

Table 7

Expected Value of Hits
With Respect to Δt

INCREMENT	PROB EXPT	n = NUMBER OF HITS				EXPECTED VALUE CORRESPOND TO THE INCREMENT
		1	2	3	4	
$0.0\Delta t$	Pr(n)	.5728	.1236	.0061	.000001	.8383
	E(n)	.5728	.2472	.0183	.000004	
$.1\Delta t$	Pr(n)	.5923	.1224	.0057	.000002	.8542
	E(n)	.5923	.2448	.0171	.000080	
$.2\Delta t$	Pr(n)	.5564	.1404	.0046	.000004	.8510
	E(n)	.5564	.2808	.0138	.000016	
$.3\Delta t$	Pr(n)	.5121	.1609	.0033	.000007	.8438
	E(n)	.5121	.3218	.0099	.000028	
$.4\Delta t$	Pr(n)	.4811	.1766	.0033	.000012	.8442
	E(n)	.4811	.3532	.0099	.000048	
$.5\Delta t$	Pr(n)	.4895	.1744	.0027	.000009	.8464
	E(n)	.4895	.3488	.0081	.000036	
$.6\Delta t$	Pr(n)	.5207	.1581	.0035	.000006	.8474
	E(n)	.5207	.3162	.0105	.000024	
$.7\Delta t$	Pr(n)	.5640	.1368	.0049	.000004	.8523
	E(n)	.5640	.2736	.0147	.000016	
$.8\Delta t$	Pr(n)	.5968	.1198	.0058	.000002	.8538
	E(n)	.5968	.2396	.0174	.000008	
$.9\Delta t$	Pr(n)	.6002	.1184	.0059	.000001	.8547
	E(n)	.6002	.2368	.0177	.000004	
$1.0\Delta t$	Pr(n)	.5723	.1314	.0050	.000005	.8501
	E(n)	.5723	.2328	.0150	.000020	

NOTE:
Pr(n) : Probability (n Hits)
E(n) : Expected Value (n Hits)

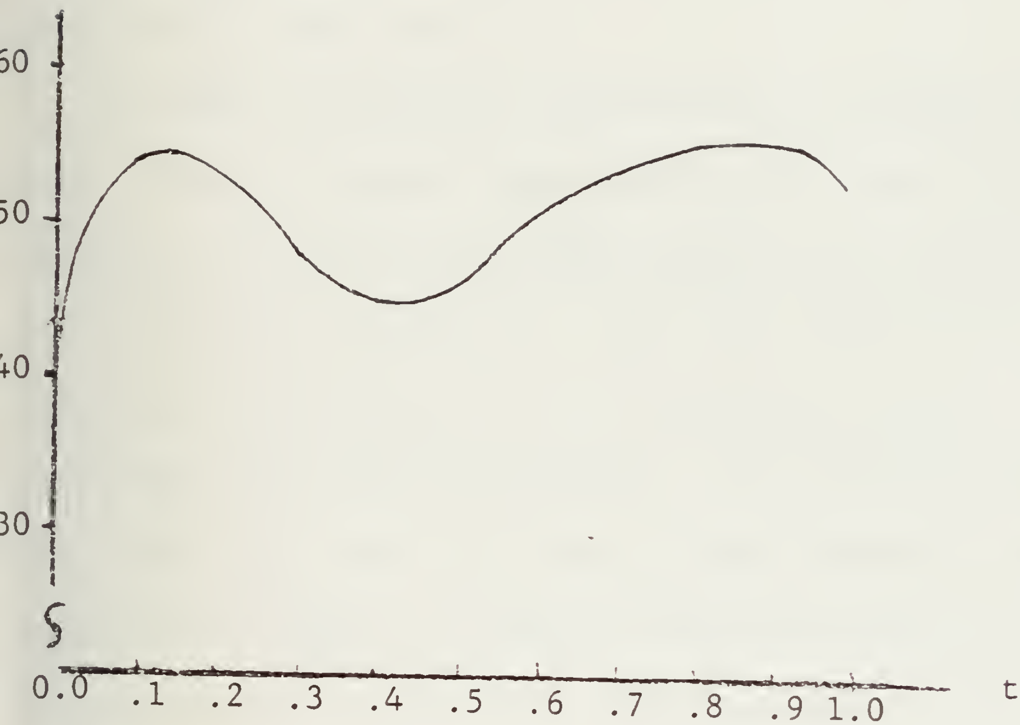

Figure 12: Expected Number of Hits with Respect to t

V. CONCLUSION AND RECOMMENDATION

The model developed in Section III is based on certain assumptions. These create idealized conditions but should still be indicative of real situations and capable of yeilding useful results.

As shown in the last illustration of the model the time intervals between impact bullets are equal since the velocity of the bullet and the aircraft are assumed constant along the trajectory. In a real situation the velocity of the bullet will be decreasing because of air resistence and earth gravitation. Since we are concerned only with small arms which have a short effective range, ignoring the effect of gravity should not give a significantly different result. On the other hand by using the correct lead angle increases the probability of hit relative to that without lead angle. If the gun is re-aimed after every round, this reduces the errors.

Another example shows that the hit probability for the first bullet is high while the second and the successive bullet have low probability. These probability depend on the velocities of the aircraft and bullet and the rate of

fire. The extension of the model by taking into account the effect of gravity and wind velocity should produce better results.

Another possible extension of this model would be the generation of hit probability if the aircraft is moving tangentially to a gunner's position.

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